

Math 4200

Wednesday November 18

4.3 Integral applications of the residue theorem; we'll discuss 4.4 magic formulas for series and for certain infinite sums of analytic functions on Friday.

Announcements: We'll finish the last example on Monday, do a few more in today's notes before your quiz.

reminder: HW for Friday November 20

4.3: 1, 2, 4, 7, 10, 14, 17, 20ab. (I've swapped #7 for #6.)

There are a lot of good worked examples in the text. We discuss related examples today and in Monday's notes.

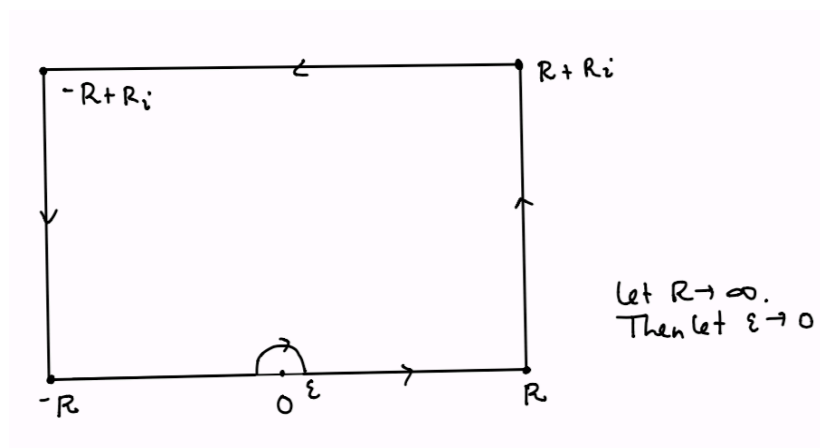
Example (Relates to homework problem 4.3.2). Show

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2},$$

using

$$\int_{\gamma_{\epsilon, R}} \frac{e^{iz}}{z} dz$$

Note, this improper integral does not converge absolutely, but converges conditionally by the alternating series test....and also, we use an interesting contour and "principal value" techniques to evaluate it.



In the previous exercise  $\frac{e^{iz}}{z}$  has a singularity at  $z=0$  even though  $\frac{\sin(x)}{x}$  is continuous at  $x=0$ . There is a general class of integrals, called *Principal Value* (or *PV*) integrals, that one can compute, even when the actual integral doesn't exist. These PV integrals are often important in e.g. physics, I think.

Def If  $f$  is continuous on  $[a, b]$  except at  $x_0 \in (a, b)$  then

$$PV\left(\int_a^b f(x) dx\right) := \lim_{\varepsilon \rightarrow 0} \left( \int_a^{x_0 - \varepsilon} f(x) dx + \int_{x_0 + \varepsilon}^b f(x) dx \right)$$

provided the limit exists.

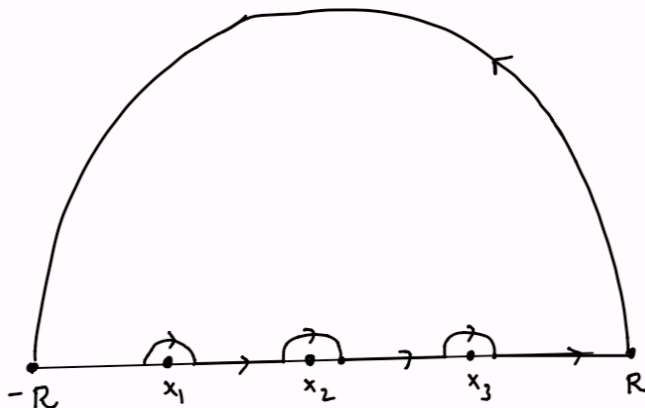
Example

$$PV\left(\int_{-1}^2 \frac{1}{x} dx\right) = \ln(2)$$

even though

$$\int_{-1}^0 \frac{1}{x} dx = -\infty, \quad \int_0^2 \frac{1}{x} dx = +\infty.$$

Using principal value ideas one can often compute  $PV\left(\int_{-\infty}^{\infty} f(x) dx\right)$  using contours like the one below. This is Proposition 4.3.11 in the text, of which our worked example was an instance.



Suggested contour for 4.3.4 (see worked example 4.3.20 in text)

$$4.3.4 \quad \int_0^{\infty} \frac{1}{1+x^6} dx = \frac{\pi}{3}$$

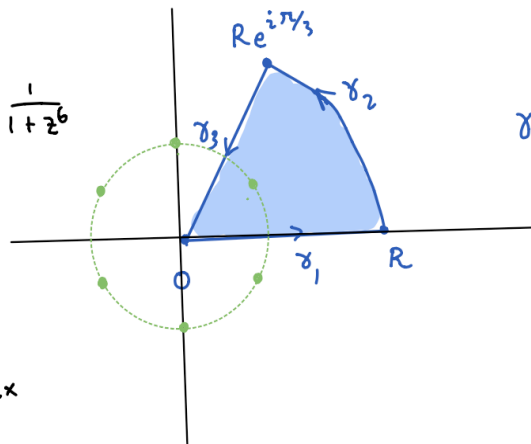
$$f(z) = \frac{1}{1+z^6}$$

(using  $\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^6} dx$  means

3 singularities in upper half plane, messy algebra)

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key point:  $\int_{\gamma_3} \frac{1}{1+z^6} dz = - \int_0^R \frac{1}{1+x^6} e^{i\pi/3} dx$   
 $z = e^{i\pi/3} x$



$$\gamma = \gamma_1 + \gamma_2 + \gamma_3$$

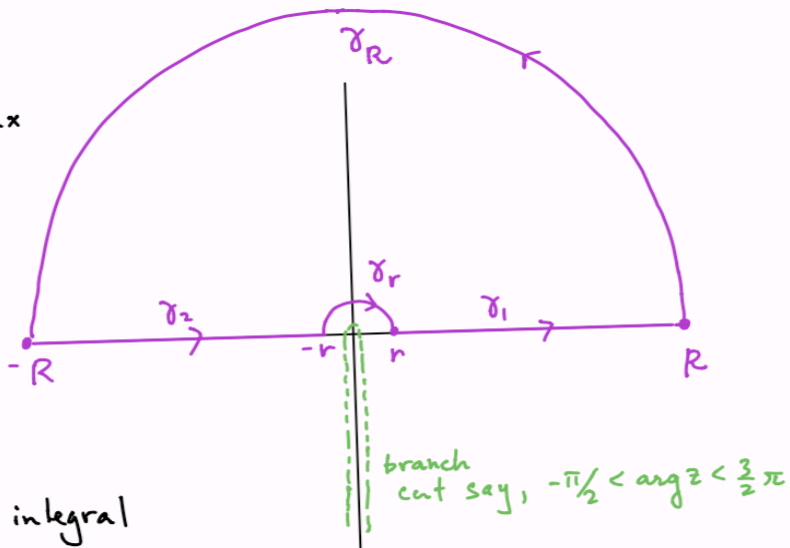
Suggested contour:

4.3.14  $\int_0^{\infty} \frac{\log x}{(x^2+1)^2} dx$

verify and use:

$$\operatorname{Re} \int_{\gamma_2} f(z) dz = \operatorname{Re} \int_{\gamma_1} f(z) dz$$

(and each limit to the integral we wish to find.)



Plus, the imaginary part of this computation will give you the value of

$$\int_0^{\infty} \frac{1}{(x^2+1)^2} dx$$